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Explicit Mumford isomorphism for hyperelliptic curves

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Abstract

Using an explicit version of the Mumford isomorphism on the moduli space of hyperelliptic curves we derive a closed formula for the Arakelov–Green function of a hyperelliptic Riemann surface evaluated at its Weierstrass points.

MSC: 14G40; 14H45; 14H55

1. Introduction

The main goal of this paper is to give a formula for the Arakelov–Green function of a hyperelliptic Riemann surface, evaluated at pairs of Weierstrass points (cf. Theorem 8.2 below). This formula generalises a result of Bost in [3] dealing with the case that the genus is 2. As an application of our formula we deduce a symmetric form of a classical identity involving Thetanullwerte and Jacobian Nullwerte, found originally by Thomae in the 19th century (cf. Theorem 9.1).

The main idea of our approach is to construct an explicit form of Mumford's isomorphism in the case of hyperelliptic curves. We recall that if $p : X \to S$ is a smooth proper curve with sheaf of relative differentials ω, one has a canonical isomorphism $\lambda_1^{\otimes 6n^2+6n+1} \xrightarrow{\sim} \lambda_n$ of invertible sheaves on S, ascribed to Mumford [21]; here n is any integer ≥ 1 and λ_n denotes the determinant sheaf $\det p_*\omega^{\otimes n}$. Later on we will find it more convenient to use a different form of Mumford's isomorphism, involving Deligne brackets, but in order to fix ideas we describe what our results mean in the present setting. Let μ_n denote the canonical trivialising section of $\lambda_n \otimes \lambda_1^{-\otimes 6n^2+6n+1}$ defined by Mumford's isomorphism. In [2], Beilinson and Schechtman give a formula for μ_n in the case where $p : X \to S$ is a family of hyperelliptic curves over the complex numbers. Their result is as follows. Let $S = \mathbb{C}^{2g+2} \setminus \{\text{diagonals}\}$ and let $p : X \to S$ be the family of hyperelliptic curves given by

$$y^2 = f_a(x) = \prod_{i=1}^{2g+2} (x - a_i), \qquad a = (a_i) \in \mathbb{C}^{2g+2}, \qquad a_i \neq a_j \text{ if } i \neq j.$$

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Put $\phi = \mathrm{d}x/y \in H^0(X, \omega)$ and consider the bases B_n of $H^0(X, \omega^{\otimes n})$ given by

$$B_1 = (\phi, x\phi, \ldots, x^{g-1}\phi),$$
$$B_n = (\phi^n, x\phi^n, \ldots, x^{n(g-1)}\phi; y\phi^n, yx\phi^n, \ldots, yx^{(n-1)(g-1)-2}\phi^n) \quad \text{for } n \geq 2.$$

Then we have

$$\mu_n = (\text{constant}) \cdot \prod_{(i,j), i \neq j} (a_i - a_j)^{n(n-1)/2} \cdot \det B_n / (\det B_1)^{\otimes 6n^2+6n+1}$$

for a running through S. The way we make Mumford's isomorphism explicit is that we are able to calculate the constant appearing in the above formula for μ_n. In fact it will follow that, up to a sign, this constant is equal to $2^{-(2g+2)n(n-1)}$.

2. Hyperelliptic curves

Even though our main result deals with hyperelliptic Riemann surfaces, we need to consider for the proof hyperelliptic curves over arbitrary base schemes. Let $g \geq 2$ be an integer, and let S be a scheme. We call a hyperelliptic curve of genus g over S any smooth, proper curve $p : X \to S$ of genus g which admits an involution σ such that for every geometric point $\bar{s}$ of S the quotient $X_{\bar{s}}/\langle\sigma\rangle$ is isomorphic to $\mathbb{P}^1_{k(\bar{s})}$. Once such an involution exists, it is unique; this is well-known for $S = \mathrm{Spec}(k)$ with k an algebraically closed field, and follows for the general case by the fact that $\mathrm{Aut}_S(X)$ is unramified over S. If $p : X \to S$ is a hyperelliptic curve, we call σ the hyperelliptic involution of X/S. Here are some facts which will be useful later on.

Proposition 2.1. *The quotient map $X \to X/\langle\sigma\rangle$ is a finite, faithfully flat S-morphism of degree 2 onto a smooth, proper S-curve of genus 0. If $X/\langle\sigma\rangle/S$ admits a section, then $X/\langle\sigma\rangle$ is S-isomorphic to $\mathbb{P}(V)$ for some locally free sheaf V on S of rank 2.*

Proof. See [18], Proposition 3.3 and Theorem 5.5. □

Let ω be the sheaf of relative differentials of X/S.

Proposition 2.2. *The image of the canonical morphism $\pi : X \to \mathbb{P}(p_*\omega)$ is a smooth, proper S-curve of genus 0. Its formation commutes with base change. Moreover, there exists a closed embedding $j : X/\langle\sigma\rangle \hookrightarrow \mathbb{P}(p_*\omega)$ such that $\pi = j \circ h$; here h is the quotient map $X \to X/\langle\sigma\rangle$.*

Proof. See [18], Lemma 5.7 and Theorem 5.5. □

The action of σ has a fixed point subscheme on X, which we denote by W. We call this scheme the Weierstrass subscheme of X. It is the closed subscheme defined locally on an affine open subscheme $U = \mathrm{Spec}(R)$ by the ideal generated by the set $\{r - \sigma(r) | r \in R\}$.

Proposition 2.3. *The Weierstrass subscheme W of X/S is the subscheme associated to a relative Cartier divisor on X. It is finite and flat over S of degree $2g+2$, and its formation commutes with base change. Furthermore, it is étale over a point $s \in S$ if and only if the residue characteristic of s is not equal to 2.*

Proof. See [18], Section 6. □

Example 2.4. Consider the proper, flat genus 2 curve $p : X \to S = \mathrm{Spec}(R)$ with $R = \mathbb{Z}[1/5]$ given by the affine equation $y^2 + x^3 y = x$. One may check that it has good reduction everywhere, and it follows that $p : X \to S$ is a hyperelliptic curve. Over the ring $R' = R[\zeta_5, \sqrt[5]{2}]$ it acquires six σ-invariant sections with one given by $x = 0$ and the others given by $x = -\zeta_5^k \sqrt[5]{4}$ for $k = 1, \ldots, 5$. The Weierstrass subscheme of X'/R' is supported on the images of these sections. It is clear that they do not meet over points of residue characteristic $\neq 2$, which verifies that indeed the Weierstrass subscheme is étale over such points. Over a prime of characteristic 2, all σ-invariant sections meet in the point given by $x = 0$. The quotient map $X_{\mathbb{F}_2} \to X_{\mathbb{F}_2}/\langle\sigma\rangle \cong \mathbb{P}^1_{\mathbb{F}_2}$ is ramified only in this point.

Remark 2.5. In general, if S is the spectrum of a field of characteristic 2, then the quotient map $X \to X/\langle\sigma\rangle$ ramifies in at most $g+1$ distinct points.

3. A canonical trivialising section of $\lambda_1^{\otimes 8g+4}$

In this section we study the invertible sheaf $\lambda_1 = \det p_*\omega$ for a hyperelliptic curve $p : X \to S$. The following proposition is perhaps well-known.

Proposition 3.1. *Suppose that S is a regular integral scheme of generic characteristic $\neq 2$ and let $p : X \to S$ be a hyperelliptic curve of genus $g \geq 2$. Then the invertible sheaf $\lambda_1^{\otimes 8g+4}$ has a canonical trivialising section Λ. In the case that $S = \mathrm{Spec}(R)$ and that X has an open subscheme $U = \mathrm{Spec}(E)$ with $E = A[y]/(y^2 + ay + b)$, where $A = R[x]$ and $a, b \in A$, one can write*

$$\Lambda = (2^{-(4g+4)} \cdot D)^g \cdot \left(\frac{\mathrm{d}x}{2y+a} \wedge \cdots \wedge \frac{x^{g-1}\mathrm{d}x}{2y+a} \right)^{\otimes 8g+4},$$

where D is the discriminant in R of the polynomial $a^2 - 4b$ in $R[x]$.

For convenience, we give here the proof; most parts of the argument are taken from [16], Section 6. The statement of Lemma 3.4 will be of importance again in the proof of Proposition 4.1. We start by considering hyperelliptic curves $p : X \to S$ of genus $g \geq 2$ with $S = \mathrm{Spec}(R)$ where R is a discrete valuation ring, say with residue field k and with quotient field K, which we assume to be of characteristic $\neq 2$. The canonical quotient map $R \to k$ will be denoted by $r \mapsto \bar{r}$.

Lemma 3.2 (*Cf. [16], Lemma 6.1*). *After a finite étale surjective base change with a discrete valuation ring R' dominating R, the scheme $X' = X \times_R R'$ can be covered by open affine subschemes of the shape $U \cong \mathrm{Spec}(E)$ with $E = A[y]/(y^2 + ay + b)$, where $A = R'[x]$ and $a, b \in A$, such that the polynomials $a^2 - 4b$ in $K'[x]$ are separable of degree $2g+2$ and such that $\deg a \leq g+1$ and $\deg b \leq 2g+2$. For the reduced polynomials $\overline{a}, \overline{b} \in k'[x]$ one always has $\deg \overline{a} = g+1$ or $\deg \overline{b} \geq 2g+1$.*

Proof. Locally in the étale topology, any smooth morphism has a section, and hence by Proposition 2.1 after a finite étale surjective base change with a discrete valuation ring R' dominating R, one obtains by taking the quotient under σ a finite faithfully flat R'-morphism $h' : X' \to \mathbb{P}^1_{R'}$ of degree 2. Choose a point $\infty \in \mathbb{P}^1_{K'}$ such that $X'_{K'} \to \mathbb{P}^1_{K'}$ is unramified above ∞, and let x be a coordinate on $V = \mathbb{P}^1_{K'} - \{\infty\}$. We can then describe $U = h'^{-1}(V)$ as $U \cong \mathrm{Spec}(E)$ with $E = A[y]/(y^2 + ay + b)$ where $A = R'[x]$ and $a, b \in A$. If we assume the degree of a to be minimal, we have $\deg a \leq g+1$ and $\deg b \leq 2g+2$. By Proposition 2.3, the Weierstrass subscheme W of X'/S' is finite and flat over S' of degree $2g+2$. By definition, the ideal of W is generated by $y - \sigma(y) = 2y + a$ on U. Note that $(2y+a)^2 = a^2 - 4b$, which defines the norm under h' of W in $\mathbb{P}^1_{R'}$. Since this norm is also finite and flat of degree $2g+2$ over B', and since W is entirely supported in U by our choice of ∞, we obtain that $\deg(a^2 - 4b) = 2g+2$. Since the norm of $W \times_{R'} K'$ in $\mathbb{P}^1_{K'}$ is étale over K' by Proposition 2.3, the polynomial $a^2 - 4b$ in $K'[x]$ is separable. Consider finally the reduced polynomials $\overline{a}, \overline{b} \in k'[x]$. Regarding y as an element of $k'(X'_{k'})$, we have $\mathrm{div}(y) \geq -\min(\deg \overline{a}, \frac{1}{2}\deg \overline{b}) \cdot h'^*(\overline{\infty})$ by the equation for y. On the other hand it follows from the theorem of Riemann–Roch that y has a pole at both points of $h'^*(\overline{\infty})$ of order strictly larger than g. This gives the last statement of the lemma. □

Lemma 3.3 (*Cf. [16], Proposition 6.2*). *Suppose we have on X an open affine subscheme $U \cong \mathrm{Spec}(E)$ as in Lemma 3.2. Then the differentials $x^i \mathrm{d}x/(2y+a)$ for $i = 0, \ldots, g-1$ are nowhere vanishing on U and extend to regular global sections of the sheaf of relative differentials ω of X/S.*

Proof. Let F be the polynomial $y^2 + a(x)y + b(x) \in A[y]$, and let F_x and F_y be its derivatives with respect to x and y, respectively. It is readily verified that the morphism $\Omega_{E/R} = (E\mathrm{d}x + E\mathrm{d}y)/(F_x\mathrm{d}x + F_y\mathrm{d}y) \to E$ given by $\mathrm{d}x \mapsto F_y, \mathrm{d}y \mapsto -F_x$, is an isomorphism of E-modules. This gives that the differentials $x^i\mathrm{d}x/(2y+a)$ for $i = 0, \ldots, g-1$ are nowhere vanishing on U. For the second part of the lemma, it suffices to show that the differentials $x^i\mathrm{d}x/(2y+a)$ for $i = 0, \ldots, g-1$ on the generic fiber U_K extend to global sections of $\Omega^1_{X_K/K}$—but this is well-known to be true. □

Lemma 3.4 (*Cf. [16], Proposition 6.3*). *Suppose we have on X an open affine subscheme $U \cong \mathrm{Spec}(E)$ as in Lemma 3.2. Let D be the discriminant in K of the polynomial $f = a^2 - 4b$ in $K[x]$. Then the modified discriminant $2^{-(4g+4)} \cdot D$ is a unit of R.*

Proof. In the case that the characteristic of k is $\neq 2$, this is not hard to see: we know that $W \times_R k$ is étale of degree $2g+2$ by Proposition 2.3, and hence f remains separable of degree $2g+2$ in $k[x]$ under the reduction map. So let us assume from now on that the characteristic of k equals 2. If B is any domain, and if $P(T) = \sum_{i=0}^{n} u_i T^i$ and $Q(T) = \sum_{i=0}^{m} v_i T^i$ are two polynomials in $B[T]$, we denote by $R_T^{n,m}(P,Q)$ the resultant in B of P and Q. It satisfies the following property: suppose that at least one of u_n, v_m is non-zero, and that B is in fact a field. Then $R_T^{n,m}(P,Q) = 0$ if and only if P and Q have a root in common in an extension field of B. Let F be the polynomial $y^2 + a(x)y + b(x)$ in $A[y]$ with $A = R[x]$, and let F_x and F_y be its derivatives with respect to x and y, respectively. We set $Q = R_y^{2,1}(F, F_x)$ and $P = R_y^{2,1}(F, F_y)$ which is $4b - a^2 = -f$. Let $H \in R$ be the leading coefficient of P, and put $\Delta = 2^{-(4g+4)} \cdot D$. A calculation (for which see for instance [17], Section 1) shows that $R_x^{2g+2,4g+2}(P,Q) = (H \cdot \Delta)^2$. We can read this equation as a formal identity between certain universal polynomials in the coefficients of $a(x)$ and $b(x)$. Doing so, we may conclude that $\Delta \in R$ and that H^2 divides $R_x^{2g+2,4g+2}(P,Q)$ in R. To show that Δ is in fact a unit, we distinguish two cases. First we assume that $\overline{H} \neq 0$. Then $\deg \overline{P} = 2g+2$ and again a calculation shows that $R_x^{2g+2,4g+2}(\overline{P}, \overline{Q}) = (\overline{H} \cdot \overline{\Delta})^2$. The fact that X_k is smooth implies that $R_x^{2g+2,4g+2}(\overline{P}, \overline{Q})$ is non-zero, and altogether we obtain that $\overline{\Delta}$ is non-zero. Now we assume that $\overline{H} = 0$. Then since $\overline{P} = \overline{a}^2$ we obtain that $\deg \overline{a} \leq g$ and hence $\deg \overline{P} \leq 2g$. By Lemma 3.2 we have then $2g+1 \leq \deg \overline{b} \leq 2g+2$. But then from $2\deg(y) = \deg(\overline{a}y + \overline{b})$ and $\deg(y) > g$, which holds by the theorem of Riemann–Roch, it follows that in fact $\deg \overline{b} = 2g+2$ and hence $\deg \frac{\mathrm{d}\overline{b}}{\mathrm{d}x} = 2g$. This implies that $\deg \overline{Q} = 4g$. A final calculation shows that $R_x^{2g,4g}(\overline{P}, \overline{Q}) = \overline{\Delta}^2$. Again by smoothness of X_k we may conclude that $R_x^{2g,4g}(\overline{P}, \overline{Q})$ is non-zero. This finishes the proof. $\square$

Example 3.5. Consider once more the curve over $R = \mathbb{Z}[1/5]$ given by the equation $y^2 + x^3 y = x$, cf. Example 2.4 above. In the notation from Lemma 3.2 we have $a = x^3$, $b = -x$. We compute $D = \mathrm{disc}(x^6 + 4x) = 2^{12} 5^5$ so that $\Delta = 5^5$ which is indeed a unit in R.

Proof of Proposition 3.1 (*Cf. [19], Proposition 2.7*). Again, since locally in the étale topology any smooth morphism has a section, it follows by Proposition 2.1 that after a faithfully flat base change the quotient map $X \to X/\langle\sigma\rangle$ becomes an S-morphism onto a $\mathbb{P}^1_S$. Then by Lemma 3.2 we may assume that the scheme X is covered by affine schemes $U \cong \mathrm{Spec}(E)$ with $E = A[y]/(y^2 + ay + b)$ and A a polynomial ring $R[x]$. For such an affine scheme U, consider $V = \mathrm{Spec}(A)$. In the line bundle $(\det p_* \omega_{U/V})^{\otimes 8g+4}$ we have a rational section

$$\Lambda_{U/V} = (2^{-(4g+4)} \cdot D)^g \cdot \left(\frac{\mathrm{d}x}{2y+a} \wedge \cdots \wedge \frac{x^{g-1}\mathrm{d}x}{2y+a} \right)^{\otimes 8g+4},$$

with D as in Lemma 3.4. One can check that this section does not depend on any choice of affine equation $y^2 + ay + b$ for U, and moreover, these sections coincide on overlaps. Hence they build a canonical rational section Λ of $\lambda_1^{\otimes 8g+4}$. By Lemmas 3.3 and 3.4, this Λ is a global trivialising section. The general case follows by faithfully flat descent. $\square$

4. Adjunction on the Weierstrass subscheme

In this section we recall the formalism of the Deligne bracket [5]. Using this formalism, we construct here a canonical section of a certain invertible sheaf on the base S of a hyperelliptic curve $p : X \to S$, which can be seen as a sort of residue map (as in the classical adjunction formula) for the Weierstrass subscheme of X/S.

Let's start with an arbitrary proper, flat, locally complete intersection curve $p : X \to S$. Deligne has shown that there exists a natural rule that associates to any pair (L, M) of invertible sheaves on X an invertible sheaf $\langle L, M \rangle$ on S, such that the following properties are satisfied:

(i) For invertible sheaves L_1, L_2, M_1, M_2 on X we have canonical isomorphisms
$\langle L_1 \otimes L_2, M \rangle \xrightarrow{\sim} \langle L_1, M \rangle \otimes \langle L_2, M \rangle$ and $\langle L, M_1 \otimes M_2 \rangle \xrightarrow{\sim} \langle L, M_1 \rangle \otimes \langle L, M_2 \rangle$.

(ii) For invertible sheaves L, M on X we have a canonical isomorphism $\langle L, M \rangle \xrightarrow{\sim} \langle M, L \rangle$.

(iii) The formation of the Deligne bracket commutes with base change, i.e., each cartesian diagram

$$\begin{tikzcd} X' \arrow[r, "u'"] \arrow[d, "p'"] & X \arrow[d, "p"] \\ S' \arrow[r, "u"] & S \end{tikzcd}$$

gives rise to a canonical isomorphism $u^*\langle L, M\rangle \xrightarrow{\sim} \langle u'^*L, u'^*M\rangle$.

(iv) For $P : S \to X$ a section of p and any invertible sheaf L on X we have a canonical isomorphism $P^*L \xrightarrow{\sim} \langle O_X(P), L\rangle$.

(v) (Adjunction formula) For the sheaf of relative differentials ω of p and any section $P : S \to X$ of p we have a canonical adjunction isomorphism $\langle P, \omega\rangle \xrightarrow{\sim} \langle P, P\rangle^{\otimes -1}$.

(vi) (Riemann–Roch) Let L be an invertible sheaf on X and let ω be the sheaf of relative differentials of X/S. Then we have a canonical isomorphism

$$(\det Rp_*L)^{\otimes 2} \xrightarrow{\sim} \langle L, L \otimes \omega^{\otimes -1}\rangle \otimes (\det Rp_*\omega)^{\otimes 2}$$

of line bundles on S, with $\det Rp_*$ denoting the determinant of cohomology along p.

In fact, one can put

$$\langle L, M\rangle = \det Rp_*(L \otimes M) \otimes (\det Rp_*L)^{-1} \otimes (\det Rp_*M)^{-1} \otimes (\det Rp_*\omega)$$

and then the properties (i)–(vi) can be checked one by one. Another fact that will be useful later is that if D is a relative Cartier divisor on X and if M is an invertible sheaf on X, one has a canonical isomorphism

$$\langle O_X(D), M\rangle \xrightarrow{\sim} \mathrm{Nm}_{D/S}(M\,|_D),$$

where $\mathrm{Nm}_{D/S}$ denotes the norm.

Now let $p : X \to S$ be a hyperelliptic curve. We will denote here by W the invertible sheaf associated to the relative Cartier divisor defined by the Weierstrass subscheme of X/S. This change of notation should cause no confusion. The Deligne bracket that we are interested in is $\langle W, W \otimes \omega\rangle$ and the statement that we want to prove about it is as follows.

Proposition 4.1. *Suppose that S is a regular integral scheme of generic characteristic $\neq 2$ and let B be the branch divisor of W/S. Then we have a canonical isomorphism $\langle W, W \otimes \omega\rangle \xrightarrow{\sim} O_S(B)$. Furthermore, let Ξ be the rational section of $\langle W, W \otimes \omega\rangle$ corresponding to the canonical rational section of $O_S(B)$ under this isomorphism. Then $2^{-(2g+2)} \cdot \Xi$ is a global trivialising section of $\langle W, W \otimes \omega\rangle$.*

Proof. By our remarks above, the invertible sheaf $\langle W, W \otimes \omega\rangle$ is canonically isomorphic to $\mathrm{Nm}((W \otimes \omega)|_W)$ and this, in turn, is canonically isomorphic to $\mathrm{Nm}(\omega_{W/S})$ by the adjunction formula. But the latter is the discriminant of W/S, which is canonically isomorphic to $O_S(B)$, with B the branch divisor of W/S. Now let's look at $2^{-(2g+2)} \cdot \Xi$ as in the statement of the proposition. We claim that it has neither zeroes nor poles on S. First of all we remark that it suffices to place ourselves in the situation where $S = \mathrm{Spec}(R)$ with R a discrete valuation ring whose fraction field K has characteristic $\neq 2$. Perhaps after making a faithfully flat cover we can assume that the Weierstrass subscheme is supported on $2g + 2$ sections $W_1, \ldots, W_{2g+2}$ and that the image of the canonical map $h : X \to X/\langle\sigma\rangle$ is a $\mathbb{P}^1_R$. We assume that the discrete valuation on R is normalised such that $v(K^*) = \mathbb{Z}$. The valuation $v(\Xi)$ of Ξ at the closed point s of S is then given by the sum $\sum_{k\neq l}(W_k, W_l)$ of the local intersection multiplicities (W_k, W_l) above s of pairs of sections W_k. Suppose that W_k is given by a polynomial $x - a_k$, and write a_k as a shorthand for the corresponding section of $\mathbb{P}^1_R$. By the projection formula we have for the local intersection multiplicities that $4(W_k, W_l) = (2W_k, 2W_l) = (h^*a_k, h^*a_l) = 2(a_k, a_l)$ for each $k \neq l$ hence $(W_k, W_l) = \frac{1}{2}(\alpha_k, \alpha_l)$ for each $k \neq l$. Now the local intersection multiplicity (a_k, a_l) above s on $\mathbb{P}^1_R$ is calculated to be $v(a_k - a_l)$. This gives that $v(\Xi) = \sum_{k\neq l}(W_k, W_l) = \frac{1}{2}\sum_{k\neq l} v(a_k - a_l)$. By Lemma 3.4 we have $\sum_{k\neq l} v(a_k - a_l) = (4g + 4)v(2)$ hence the valuation of $2^{-(2g+2)} \cdot \Xi$ vanishes at s, which is what we wanted. The general case follows from this by faithfully flat descent. □

5. Arakelov theory of compact Riemann surfaces

Our main result gives a relation between the Arakelov–Green function of a hyperelliptic Riemann surface, evaluated at its Weierstrass points, and the Faltings delta-invariant of that Riemann surface. We introduce these notions in the present section; for some motivating background and for more results we refer to Arakelov's original paper [1] and Faltings' paper [6].

We start by fixing a compact Riemann surface X of positive genus g. On the space $H^0(X, \omega)$ of holomorphic differential forms we have a natural hermitian inner product $(\alpha, \beta) \mapsto \frac{i}{2}\int_X \alpha \wedge \overline{\beta}$. Let $(\alpha_1, \ldots, \alpha_g)$ be an orthonormal basis for this inner product. It can be used to build a smooth real (1,1)-form on X given by $\mu = \frac{i}{2g}\sum_{k=1}^{g} \alpha_k \wedge \overline{\alpha_k}$. Obviously μ does not depend on the choice of orthonormal basis, and hence is canonical. The Arakelov–Green function of X is now the unique function $G : X \times X \to \mathbb{R}_{\geq 0}$ satisfying the following properties for all $P \in X$:

(I) The function $\log G(P, Q)$ is C^∞ for $Q \neq P$.
(II) We can write $\log G(P, Q) = \log |z_P(Q)| + f(Q)$ locally about P, where z_P is a local coordinate about P and where f is C^∞ about P.
(III) We have $\partial_Q \overline{\partial}_Q \log G(P, Q)^2 = 2\pi i \mu(Q)$ for $Q \neq P$.
(IV) We have $\int_X \log G(P, Q)\mu(Q) = 0$.

Existence and uniqueness of G are proved in [1]. By an application of Stokes' theorem one finds the symmetry relation $G(P, Q) = G(Q, P)$ for all $P, Q \in X$.

An admissible line bundle on X is a line bundle L on X together with a smooth hermitian metric on L such that the curvature form of L is a multiple of μ. Using the Arakelov–Green function, one obtains a canonical structure of admissible line bundle on line bundles of the form $O_X(P)$, with P a point on X, as follows: let s be the tautological section of $O_X(P)$, then put $\|s\|(Q) = G(P, Q)$ for any $Q \in X$. By property (III) above, the curvature form of $O_X(P)$ with this metric is equal to μ. Any other admissible metric on $O_X(P)$ is a constant multiple of the canonical metric; furthermore we get canonical metrics on line bundles of the form $O_X(D)$ with D a divisor on X by taking tensor products. A very important admissible line bundle is the line bundle ω of holomorphic differentials, endowed with its Arakelov metric $\| \cdot \|_{\mathrm{Ar}}$; this metric can be defined by insisting that for every P on X, the residue isomorphism $\omega(P)[P] = (\omega \otimes O_X(P))[P] \xrightarrow{\sim} \mathbb{C}$ is an isometry, with $\mathbb{C}$ having its standard euclidean metric. It is proved in [1] that this metric is indeed admissible.

For any admissible line bundle L on X, Faltings has defined a certain metric on the determinant of cohomology $\lambda(L) = \det H^0(X, L) \otimes \det H^1(X, L)^\vee$ of the underlying line bundle (cf. [6], Theorem 1). We do not recall the definition, but mention only that for $L = \omega$, the metric on $\lambda(L) \cong \det H^0(X, \omega)$ is the one given by the inner product $(\alpha, \beta) \mapsto \frac{i}{2}\int_X \alpha \wedge \overline{\beta}$ on $H^0(X, \omega)$. It turns out that the Faltings metric on the determinant of cohomology can be made explicit using theta functions. Let $\mathcal{H}_g$ be the Siegel upper half space of complex symmetric g-by-g-matrices with positive definite imaginary part. Let $\tau \in \mathcal{H}_g$ be a period matrix associated to a symplectic basis of $H_1(X, \mathbb{Z})$ and consider the complex torus $J_\tau(X) = \mathbb{C}^g/\mathbb{Z}^g + \tau\mathbb{Z}^g$ associated to τ. On $\mathbb{C}^g$ one has the Riemann theta function $\vartheta(z; \tau) = \sum_{n \in \mathbb{Z}^g} \exp(\pi i {}^t n \tau n + 2\pi i {}^t n z)$, giving rise to an effective divisor Θ_0 and a line bundle $O(\Theta_0)$ on $J_\tau(X)$. Now consider on the other hand the set $\mathrm{Pic}_{g-1}(X)$ of divisor classes of degree $g-1$ on X. It comes with a canonical subset Θ given by the classes of effective divisors. By the theorem of Abel–Jacobi–Riemann there is a canonical bijection $u : \mathrm{Pic}_{g-1}(X) \xrightarrow{\sim} J_\tau(X)$ mapping Θ onto Θ_0. As a result, we can equip $\mathrm{Pic}_{g-1}(X)$ with the structure of a compact complex manifold, together with a divisor Θ and a line bundle $O(\Theta)$.

The function ϑ is not well-defined on $\mathrm{Pic}_{g-1}(X)$ or $J_\tau(X)$. We can remedy this by putting $\|\vartheta\|(z; \tau) = (\det \mathrm{Im}\ \tau)^{1/4} \exp(-\pi {}^t y (\mathrm{Im}\ \tau)^{-1} y)|\vartheta(z; \tau)|$, with $y = \mathrm{Im}\ z$. One can check that $\|\vartheta\|$ descends to a function on $J_\tau(X)$. By our identification $\mathrm{Pic}_{g-1}(X) \xrightarrow{\sim} J_\tau(X)$ we obtain $\|\vartheta\|$ as a function on $\mathrm{Pic}_{g-1}(X)$. It can be checked that this function is independent of the choice of τ. Note that $\|\vartheta\|$ gives a canonical way to put a metric on the line bundle $O(\Theta)$ on $\mathrm{Pic}_{g-1}(X)$.

For any line bundle L of degree $g-1$ there is a canonical isomorphism $\lambda(L) \xrightarrow{\sim} O(-\Theta)[L]$, the fiber of $O(-\Theta)$ at the class in $\mathrm{Pic}_{g-1}(X)$ determined by L. Faltings proves in [6] that when we give both sides the metrics discussed above, the norm of this isomorphism is a constant independent of L; he writes it as $\mathrm{e}^{\delta(X)/8}$. The $\delta(X)$ appearing here is the celebrated Faltings delta-invariant of X. An important formula relating G and δ follows from these considerations. Again, let $(\alpha_1, \ldots, \alpha_g)$ be an orthonormal basis of $H^0(X, \omega)$, and let $P_1, \ldots, P_g, Q$ be distinct points on X. Then the formula

$$\|\vartheta\|(P_1 + \cdots + P_g - Q) = \mathrm{e}^{-\delta(X)/8} \cdot \frac{\| \det \alpha_k(P_l)\|_{\mathrm{Ar}}}{\prod_{k<l} G(P_k, P_l)} \cdot \prod_{k=1}^{g} G(P_k, Q) \qquad (*)$$

holds (see [6], p. 402). An important counterpart to this formula was derived by Guàrdia [8]; we will state a special case of his formula in Section 9 below.

It is possible for L, M admissible line bundles on X, to endow the invertible sheaves (vector spaces) $\langle L, M\rangle$ with natural metrics (called Arakelov metrics here), such that all isomorphisms in (i)–(v) of Section 4 above become isometries. In particular if $L = O_X(P)$ and $M = O_X(Q)$ then $\langle L, M\rangle$ has a certain tautological section $\langle s_P, s_Q\rangle$ whose norm is just $G(P, Q)$. Faltings' metric on the determinant of cohomology has the property that for all admissible line bundles L and with the canonical Arakelov metrics on all Deligne brackets of pairs of admissible line bundles, the Riemann–Roch isomorphism (vi) is always an isometry.

6. Self-intersection of the sheaf of relative differentials

The purpose of this section is to prove the following proposition.

Proposition 6.1. *Let $p : X \to S$ be a hyperelliptic curve of genus $g \geq 2$ with sheaf of relative differentials ω. If P, Q are σ-invariant sections of p then we have a canonical isomorphism*

$$\langle \omega, \omega\rangle \xrightarrow{\sim} \langle P, Q\rangle^{\otimes -4g(g-1)}$$

of invertible sheaves on S, compatible with base change. If $B = \mathrm{Spec}(\mathbb{C})$, then the above isomorphism is an isometry, provided both sides are endowed with their canonical Arakelov metrics.

We need one lemma, which is a generalisation of Proposition 1 in Section 1.1 of [4].

Lemma 6.2. *Let $p : X \to S$ be a hyperelliptic curve of genus $g \geq 2$ with sheaf of relative differentials ω. For any σ-invariant section $P : S \to X$ of p we have a unique isomorphism*

$$\omega \xrightarrow{\sim} O_X((2g-2)P) \otimes p^*\langle P, P\rangle^{\otimes -(2g-1)}$$

that induces, by pulling back along P, the adjunction isomorphism $\langle P, \omega\rangle \xrightarrow{\sim} \langle P, P\rangle^{\otimes -1}$. The formation of this isomorphism commutes with base change. If $B = \mathrm{Spec}(\mathbb{C})$, then the above isomorphism is an isometry, provided both sides are endowed with their canonical Arakelov metrics.

Proof. First of all, let P be *any* section of p. Let $h : X \to X/\langle\sigma\rangle$ be the canonical map. We recall that $X/\langle\sigma\rangle$ is a smooth, proper S-curve of genus 0. Let $q : X/\langle\sigma\rangle \to S$ be its structure morphism. By composing P with h we obtain a section Q of q, and as a result we can write $X/\langle\sigma\rangle \cong \mathbb{P}(V)$ for some locally free sheaf V of rank 2 on B. On the other hand, consider the canonical morphism $\pi : X \to \mathbb{P}(p_*\omega)$. This gives us a natural isomorphism $\omega \cong \pi^*(O_{\mathbb{P}(p_*\omega)}(1))$. Let $j : X/\langle\sigma\rangle \hookrightarrow \mathbb{P}(p_*\omega)$ be the closed embedding given by Proposition 2.2. Passing to a faithfully flat cover, we get that j is isomorphic to a Veronese embedding $\mathbb{P}^1 \hookrightarrow \mathbb{P}^{g-1}$ (cf. [18], Remark 5.11), and hence, using a faithfully flat descent argument, one has a natural isomorphism $j^*(O_{\mathbb{P}(p_*\omega)}(1)) \cong O_{\mathbb{P}(V)}(g-1)$. By well-known properties of projective bundles there exists a unique invertible sheaf L on S such that $O_{\mathbb{P}(V)}(g-1) \cong O_{\mathbb{P}(V)}((g-1)\cdot Q) \otimes q^*L$. By pulling back along h, we find a natural isomorphism $\omega \xrightarrow{\sim} O_X((g-1)\cdot(P+\sigma(P))) \otimes p^*L$. In the special case where P is σ-invariant, this leads to a natural isomorphism $\omega \xrightarrow{\sim} O_X((2g-2)P) \otimes p^*L$. Pulling back along P we find that $L \cong \langle\omega, P\rangle \otimes \langle P, P\rangle^{\otimes -(2g-2)}$ and with the adjunction isomorphism $\langle P, P\rangle \cong \langle -P, \omega\rangle$ then finally $L \cong \langle P, P\rangle^{\otimes -(2g-1)}$. It is now clear that we have an isomorphism $\omega \xrightarrow{\sim} O_X((2g-2)P) \otimes p^*\langle P, P\rangle^{\otimes -(2g-1)}$ that induces by pulling back along P an isomorphism $\langle P, \omega\rangle \xrightarrow{\sim} \langle P, P\rangle^{\otimes -1}$. Possibly after multiplying with a unique global section of O_S^*, we can establish that the latter isomorphism be the adjunction isomorphism. The commutativity with base change is clear from the general base change properties of ω and of the Deligne bracket. If $B = \mathrm{Spec}(\mathbb{C})$ then our isomorphism multiplies the Arakelov metrics by a constant because both sides are admissible and hence have the same curvature form. As the adjunction isomorphism is an isometry, our isomorphism is an isometry at P, and hence everywhere. □

The proof of Proposition 6.1 is strongly inspired by the proof of Proposition 2 in Section 1.2 of [4].

Proof of Proposition 6.1. By Lemma 6.2, we have canonical isomorphisms

$$\omega \xrightarrow{\sim} O_X((2g-2)P) \otimes p^*\langle P, P\rangle^{\otimes -(2g-1)}$$

and

$$\omega \xrightarrow{\sim} O_X((2g-2)Q) \otimes p^*\langle Q, Q\rangle^{\otimes -(2g-1)}.$$

It follows that $O_X((2g-2)(P-Q))$ comes from the base, say $O_X((2g-2)(P-Q)) \xrightarrow{\sim} p^*L$, and hence

$$\langle (2g-2)(P-Q), P-Q\rangle \xrightarrow{\sim} P^*p^*L \otimes Q^*p^*L^{\otimes -1} = L \otimes L^{\otimes -1}$$

is canonically trivial on S. Expanding, we get a canonical isomorphism

$$\langle P, P\rangle^{\otimes 2g-2} \otimes \langle Q, Q\rangle^{\otimes 2g-2} \xrightarrow{\sim} \langle P, Q\rangle^{\otimes 2(2g-2)}$$

of invertible sheaves on S. Expanding next the right hand member of the canonical isomorphism

$$\langle \omega, \omega\rangle \xrightarrow{\sim} \langle O_X((2g-2)P) \otimes p^*\langle P, P\rangle^{\otimes -(2g-1)}, O_X((2g-2)Q) \otimes p^*\langle Q, Q\rangle^{\otimes -(2g-1)}\rangle$$

gives the result. The commutativity with base change is clear. The statement on the norm follows since all the isomorphisms above are isometries. This is clear from Lemma 6.2, except possibly for the isomorphism $\langle P, P\rangle^{\otimes 2g-2} \otimes \langle Q, Q\rangle^{\otimes 2g-2} \xrightarrow{\sim} \langle P, Q\rangle^{\otimes 2(2g-2)}$. But here the statement follows since $O_X((2g-2)(P-Q))$ comes from the base, and hence its Arakelov metric is constant. By pulling back along P and along Q this constant is cancelled away, resulting in the trivial metric on $\langle (2g-2)(P-Q), P-Q\rangle$ under its canonical trivialisation. □

7. Explicit Mumford isomorphism

Let $p : X \to S$ be a smooth, proper curve with sheaf of relative differentials ω. As was mentioned in the Introduction, we have a canonical isomorphism $\lambda_1^{\otimes 6n^2+6n+1} \xrightarrow{\sim} \lambda_n$ for any integer $n \geq 1$, where λ_n is defined to be the determinant sheaf $\det p_*\omega^{\otimes n}$. By Serre duality, this sheaf equals the determinant of cohomology $\det Rp_*\omega^{\otimes n}$ of $\omega^{\otimes n}$. Taking $n = 2$ and applying the Riemann–Roch isomorphism of Section 4 we obtain a canonical isomorphism

$$\text{(M)} \qquad \mu : \lambda_1^{\otimes 12} \xrightarrow{\sim} \langle \omega, \omega\rangle.$$

We have the following result on the norm of μ.

Proposition 7.1 (*Faltings [6], Moret-Bailly [20]*). *Assume that $S = \mathrm{Spec}(\mathbb{C})$ and endow both sides of the isomorphism (M) with their canonical Arakelov metrics. Let g be the genus of X. Then the norm of μ is equal to $(2\pi)^{-4g}\mathrm{e}^{\delta(X)}$ where $\delta(X)$ is the Faltings delta-invariant of X as in Section 5.*

Now let's consider the case that $p : X \to S$ is a hyperelliptic curve. Using the results of Section 4 we can identify a certain power of $\langle \omega, \omega\rangle$ with a certain power of $\langle W, W \otimes \omega\rangle$, where W is the invertible sheaf associated to the Weierstrass subscheme as in Section 4. Applying the Mumford isomorphism (M), one can thus identify a certain power of λ_1 with a certain power of $\langle W, W \otimes \omega\rangle$. The interesting point is that in this way one can identify a certain power of the canonical section Λ, on the one hand, with a certain power of the canonical section $2^{-(2g+2)} \cdot \Xi$, on the other. More precisely, one has the following result.

Theorem 7.2. *Let $p : X \to S$ be a hyperelliptic curve of genus $g \geq 2$ with S a regular integral scheme of generic characteristic $\neq 2$ and suppose that there exist $2g+2$ distinct σ-invariant sections. Then one has a canonical isomorphism*

$$\lambda_1^{\otimes 12(8g+4)(4g^2+6g+2)} \xrightarrow{\sim} \langle W, W \otimes \omega\rangle^{\otimes -4g(g-1)(8g+4)}.$$

This isomorphism maps $\Lambda^{\otimes 12(4g^2+6g+2)}$ to $(2^{-(2g+2)} \cdot \Xi)^{\otimes -4g(g-1)(8g+4)}$, up to a sign. In the case that $S = \mathrm{Spec}(\mathbb{C})$, the isomorphism has norm $\left((2\pi)^{-4g}\mathrm{e}^{\delta(X)}\right)^{(8g+4)(4g^2+6g+2)}$, if both sides are equipped with their canonical Arakelov metrics.

Proof. Let P, Q be distinct σ-invariant sections of $X \to S$. By Proposition 6.1 one has a canonical isomorphism $\langle \omega, \omega\rangle \xrightarrow{\sim} \langle P, Q\rangle^{\otimes -4g(g-1)}$, which is an isometry for the canonical Arakelov metrics. Using the adjunction formula for the Deligne bracket one obtains from this a canonical isomorphism $\langle \omega, \omega\rangle^{\otimes 4g^2+6g+2} \xrightarrow{\sim} \langle W, W \otimes \omega\rangle^{\otimes -4g(g-1)}$

which is again an isometry for the Arakelov metrics. Applying the Mumford isomorphism (M) one gets a canonical isomorphism $\lambda_1^{\otimes 12(4g^2+6g+2)} \xrightarrow{\sim} \langle W, W \otimes \omega\rangle^{\otimes -4g(g-1)}$ having norm $((2\pi)^{-4g} e^{\delta(X)})^{4g^2+6g+2}$ by Proposition 7.1. The required isomorphism and the statement on its norm follow from this by raising to the $(8g+4)$-th power. Now as to the sections on both sides, recall from Proposition 3.1 that Λ is a canonical trivialising section of $\lambda_1^{\otimes 8g+4}$. On the other hand, by Proposition 4.1 we have that $2^{-(2g+2)} \cdot \Xi$ is a canonical trivialising section of $\langle W, W \otimes \omega\rangle$. The proof of the theorem is therefore completed by the following proposition. □

Proposition 7.3 (*Cf. [11], Lemma 2.1*). *Let $\mathcal{I}_g$ be the stack of hyperelliptic curves of genus $g \geq 2$. Then $H^0(\mathcal{I}_g, \mathbb{G}_m) = \{-1, +1\}$.*

Proof. We note that we can describe $\mathcal{I}_g \otimes \mathbb{C}$ as the space of $(2g+2)$-tuples of distinct points on $\mathbb{P}^1$ modulo projective equivalence. More precisely one has $\mathcal{I}_g \otimes \mathbb{C} = ((\mathbb{P}^1 \setminus \{0, 1, \infty\})^{2g-1} \setminus \{\text{diagonals}\})/S_{2g+2}$ where S_{2g+2} is the symmetric group acting by permutation on $2g+2$ points on $\mathbb{P}^1$. According to Theorem 10.6 of [10] the first homology of $(\mathbb{P}^1 \setminus \{0, 1, \infty\})^{2g-1} \setminus \{\text{diagonals}\}$ is isomorphic to the irreducible representation of S_{2g+2} corresponding to the partition $\{2g, 2\}$ of $2g+2$; in particular it does not contain a trivial representation of S_{2g+2}. This proves that $H_1(\mathcal{I}_g \otimes \mathbb{C}, \mathbb{Q})$ is trivial, and hence $H^0(\mathcal{I}_g \otimes \mathbb{C}, \mathbb{G}_m) = \mathbb{C}^*$. The statement that $H^0(\mathcal{I}_g, \mathbb{G}_m) = \{-1, +1\}$ follows from this since $\mathcal{I}_g \to \mathrm{Spec}(\mathbb{Z})$ is smooth and surjective. □

8. Arakelov–Green function at Weierstrass points

In this section we derive from Theorem 7.2 our main result, which is an expression for the Arakelov–Green function of a hyperelliptic Riemann surface, evaluated at its Weierstrass points, in terms of the discriminant of that surface and its Faltings delta-invariant. Our formula can be seen as a generalisation of a formula in Proposition 4 of [3], which deals with the special case of Riemann surfaces of genus 2.

Before we state the theorem, we need to introduce the discriminant. Let $g \geq 2$ be an integer and let again $\mathcal{H}_g$ be the Siegel upper half space. For vectors $\eta', \eta'' \in \frac{1}{2}\mathbb{Z}^g$ (viewed as column vectors) we have on $\mathbb{C}^g \times \mathcal{H}_g$ a theta function $\vartheta[\eta]$ with theta characteristic $\eta = (\eta', \eta'')$ given by

$$\vartheta[\eta](z; \tau) = \sum_{n \in \mathbb{Z}^g} \exp(\pi i^t (n+\eta')\tau(n+\eta') + 2\pi i^t (n+\eta')(z+\eta'')).$$

For a given theta characteristic η, the corresponding theta function $\vartheta[\eta](z; \tau)$ is either odd or even as a function of z. We call the theta characteristic η odd if the corresponding theta function $\vartheta[\eta](z; \tau)$ is odd, and even if the corresponding theta function $\vartheta[\eta](z; \tau)$ is even.

Now let X be a hyperelliptic Riemann surface of genus g. We fix an ordering $W_1, \ldots, W_{2g+2}$ of its Weierstrass points. As is explained in [22], Chapter IIIa, this induces a canonical symplectic basis of $H_1(X, \mathbb{Z})$. Next choose a coordinate x on $\mathbb{P}^1$ which puts W_{2g+2} at infinity. This gives us an affine equation $y^2 = f(x)$ of X, with f monic and separable of degree $2g+1$. Denote by $\mu_1, \ldots, \mu_g$ the holomorphic differentials on X given in coordinates by $\mu_1 = \mathrm{d}x/2y, \ldots, \mu_g = x^{g-1}\mathrm{d}x/2y$ and denote by $(\mu|\mu')$ the period matrix of $\mu_1, \ldots, \mu_g$ on the canonical symplectic basis of homology fixed by our ordering of the Weierstrass points. The matrix μ is invertible and we put $\tau = \mu^{-1}\mu'$. This matrix lies in $\mathcal{H}_g$ and we form from it the complex torus $J_\tau(X) = \mathbb{C}^g/\mathbb{Z}^g + \tau\mathbb{Z}^g$. Recall from Section 5 the Abel–Jacobi–Riemann map $u : \mathrm{Pic}_{g-1}(X) \xrightarrow{\sim} J_\tau(X)$ identifying the subset Θ of classes of effective divisors of degree $g-1$ with the zero locus of the Riemann theta function $\vartheta(z; \tau) = \sum_{n \in \mathbb{Z}^g} \exp(\pi i^t n\tau n + 2\pi i^t nz)$. It is well-known that this map satisfies $u([K_X - D]) = -u([D])$ for all divisors D of degree $g-1$; here K_X denotes a canonical divisor on X. We obtain a bijection

$$\{\text{classes of } D \text{ with } 2D \sim K_X\} \xrightarrow{\sim} J_\tau(X)[2]$$

and hence a bijection

$$\{\text{classes of } D \text{ with } 2D \sim K_X\} \xrightarrow{\sim} \{\text{classes mod } \mathbb{Z}^g \times \mathbb{Z}^g \text{ of theta characteristics}\}$$

given by $[D] \mapsto [(\eta', \eta'')]$ if $u([D]) = [\eta' + \tau \cdot \eta'']$ on $J_\tau(X)$. Using the Weierstrass points of X, it is easy to produce divisors D with $2D \sim K_X$ (we call such divisors semi-canonical divisors for short). Indeed, let W be any

Weierstrass point and let E be a divisor from the hyperelliptic pencil on X; then we have $2W \sim E$. But also we have $(g-1)E \sim K_X$ hence any divisor of degree $g-1$ with support on the Weierstrass points is semi-canonical.

We start here by considering semi-canonical divisors of the form $W_{i_1} + \cdots + W_{i_g} - W_{i_{g+1}}$ for some subset $\{i_1, \ldots, i_{g+1}\}$ of cardinality $g+1$ of $\{1, \ldots, 2g+2\}$. Such divisors have h^0 equal to 0, that is, they are never linearly equivalent to an effective divisor. The remarkable point is that the corresponding theta characteristic depends only on the set $\{i_1, \ldots, i_{g+1}\}$, and not on X. In other words, we find a canonical map

$$\{\text{subsets } S \text{ of } \{1, \ldots, 2g+2\} \text{ with } \#S = g+1\} \longrightarrow \{\text{classes mod } \mathbb{Z}^g \times \mathbb{Z}^g \text{ of theta characteristics}\}.$$

One can prove that this map is 2-to-1; in fact $W_{i_1} + \cdots + W_{i_g} - W_{i_{g+1}} \sim W_{i'_1} + \cdots + W_{i'_g} - W_{i'_{g+1}}$ if and only if $\{i_1, \ldots, i_{g+1}\} = \{i'_1, \ldots, i'_{g+1}\}$ or $\{i_1, \ldots, i_{g+1}\} \cup \{i'_1, \ldots, i'_{g+1}\} = \{1, \ldots, 2g+2\}$. Moreover, the theta characteristics in the image are always even. If S is any subset of $\{1, \ldots, 2g+2\}$ of cardinality $g+1$, we denote by η_S its corresponding theta characteristic. An explicit formula for this correspondence is given in [22], Chapter IIIa, where one finds much more details on what we have said above.

Let $\mathcal{S}$ be the set of subsets of $\{1, \ldots, 2g+2\}$ of cardinality $g+1$. We define on $\mathcal{H}_g$ the function

$$\varphi_g(\tau) = \prod_{S \in \mathcal{S}} \vartheta[\eta_S](0; \tau)^4.$$

According to [17], Section 3 the function $\varphi_g(\tau)$ is a modular form on $\Gamma_g(2) = \{\gamma \in \mathrm{Sp}(2g, \mathbb{Z}) \mid \gamma \equiv I_{2g} \bmod 2\}$ of weight $4r$ where $r = \binom{2g+1}{g+1}$. It generalises the usual Jacobi discriminant modular form in dimension 1. For period matrices τ which are associated as above to hyperelliptic Riemann surfaces, the values $\varphi_g(\tau)$ can be related to the discriminant of a hyperelliptic equation.

Proposition 8.1. *Let X be a hyperelliptic Riemann surface of genus $g \geq 2$. Fix an ordering $W_1, \ldots, W_{2g+2}$ of its Weierstrass points. Consider an equation $y^2 = f(x)$ for X with f monic and separable of degree $2g+1$, putting W_{2g+2} at infinity. Let μ_k for $k = 1, \ldots, g$ be the holomorphic differential on X given in coordinates by $\mu_k = x^{k-1}\mathrm{d}x/2y$ and let $(\mu|\mu')$ be the period matrix of these differentials on the canonical symplectic basis of homology determined by the chosen ordering of the Weierstrass points. Let $\tau = \mu^{-1}\mu'$, let $n = \binom{2g}{g+1}$ and let $r = \binom{2g+1}{g+1}$. Finally let D be the discriminant of f. Then the equality*

$$D^n = \pi^{4gr} (\det \mu)^{-4r} \varphi_g(\tau)$$

holds.

Proof. See [17], Proposition 3.2. $\square$

For a hyperelliptic Riemann surface X of genus $g \geq 2$ we define the Petersson norm of the modular discriminant of X to be $\|\varphi_g\|(X) = (\det \mathrm{Im}\, \tau)^{2r} |\varphi_g(\tau)|$ where τ is any period matrix for X formed on a canonical symplectic basis. It can be checked that the Petersson norm of the modular discriminant of X does not depend on the choice of this basis, and hence is a (natural and classical) invariant of X. It follows from Proposition 8.1 above that it does not vanish. Our main result is now as follows.

Theorem 8.2. *Let X be a hyperelliptic Riemann surface of genus $g \geq 2$. Let $m = \binom{2g+2}{g}$ and $n = \binom{2g}{g+1}$. Then we have*

$$\prod_{(W,W')} G(W, W')^{n(g-1)} = \pi^{-2g(g+2)m} \cdot \mathrm{e}^{-m(g+2)\delta(X)/4} \cdot \|\varphi_g\|(X)^{-\frac{3}{2}(g+1)},$$

the product running over all ordered pairs of distinct Weierstrass points of X.

Proof. We compute the norms of the sections Λ and Ξ for X (considered as a smooth, proper curve over $S = \mathrm{Spec}(\mathbb{C})$) and apply the result of Theorem 7.2. The formula then drops out. We start with Λ. As usual, we fix an ordering $W_1, \ldots, W_{2g+2}$ of the Weierstrass points of X and let $y^2 = f(x)$ with f monic and separable of degree $2g+1$ be an equation for X. A small computation shows that we may write

$$\Lambda = (2^{-(4g+4)} \cdot D)^g \left(\frac{\mathrm{d}x}{y} \wedge \cdots \wedge \frac{x^{g-1}\mathrm{d}x}{y} \right)^{\otimes 8g+4}$$

for the canonical trivialising element of $\det H^0(X, \omega)$, where D is the discriminant of f. Let μ_k for $k = 1, \ldots, g$ be the holomorphic differential on X given in coordinates by $\mu_k = x^{k-1}\mathrm{d}x/2y$ and let $(\mu|\mu')$ be the period matrix of these differentials on the canonical symplectic basis of homology determined by the chosen ordering of the Weierstrass points. Let $\tau = \mu^{-1}\mu'$, let $r = \binom{2g+1}{g+1}$ and put $\Delta_g = 2^{-(4g+4)n} \cdot \varphi_g$. We can then write, by Proposition 8.1,

$$\begin{aligned}
\Lambda^{\otimes n} &= (2^{-(4g+4)} \cdot D)^{gn} \left(\frac{\mathrm{d}x}{y} \wedge \cdots \wedge \frac{x^{g-1}\mathrm{d}x}{y} \right)^{\otimes (8g+4)n} \\
&= 2^{-(4g+4)gn} \pi^{4g^2 r} (\det \mu)^{-4gr} \varphi_g(\tau)^g \left(\frac{\mathrm{d}x}{y} \wedge \cdots \wedge \frac{x^{g-1}\mathrm{d}x}{y} \right)^{\otimes (8g+4)n} \\
&= (2\pi)^{4g^2 r} (\det \mu)^{-4gr} \Delta_g(\tau)^g \left(\frac{\mathrm{d}x}{2y} \wedge \cdots \wedge \frac{x^{g-1}\mathrm{d}x}{2y} \right)^{\otimes (8g+4)n} .
\end{aligned}$$

Let $J_\tau(X) = \mathbb{C}^g/\mathbb{Z}^g + \tau\mathbb{Z}^g$, and let j be the canonical isomorphism $\det H^0(X, \omega) \xrightarrow{\sim} \det H^0(J_\tau(X), \omega)$. Letting $z_1, \ldots, z_g$ be the standard euclidean coordinates on $J_\tau(X)$ we obtain from the above calculation

$$j^{\otimes (8g+4)n}(\Lambda^{\otimes n}) = (2\pi)^{4g^2 r} \Delta_g(\tau)^g (dz_1 \wedge \cdots \wedge dz_g)^{\otimes (8g+4)n}.$$

It follows that the norm of Λ satisfies

$$\|\Lambda\|^n = (2\pi)^{4g^2 r} \|\Delta_g\|(X)^g,$$

where $\|\Delta_g\|(X) = 2^{-(4g+4)n} \cdot \|\varphi_g\|(X)$; indeed, by definition the norm of $\mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_g$ is $\|\mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_g\| = \sqrt{\det \operatorname{Im} \tau}$. Now we consider the section Ξ. It has norm

$$\|\Xi\| = \prod_{(W,W')} G(W, W')$$

with the product running over all ordered pairs of distinct Weierstrass points of X. Applying Theorem 7.2 we have

$$((2\pi)^{-4g} \mathrm{e}^{\delta(X)})^{(8g+4)(4g^2+6g+2)} \cdot \|\Lambda\|^{12(4g^2+6g+2)} = \|2^{-(2g+2)} \cdot \Xi\|^{-4g(g-1)(8g+4)}.$$

Plugging in the formulas for $\|\Lambda\|$ and $\|\Xi\|$ that we just gave one obtains the required formula. □

Remark 8.3. In [14] we constructed two natural invariants $S(X)$ and $T(X)$ of compact Riemann surfaces X, related to the delta-invariant by the formula $\mathrm{e}^{\delta(X)/4} = S(X)^{-(g-1)/g^2} \cdot T(X)$. Putting $G' = S(X)^{-1/g^3} \cdot G$ the formula in Theorem 8.2 can be rewritten as

$$\prod_{(W,W')} G'(W, W')^{n(g-1)} = \pi^{-2g(g+2)m} \cdot T(X)^{-(g+2)m} \cdot \|\varphi_g\|(X)^{-\frac{3}{2}(g+1)}.$$

In this form our formula is instrumental in the paper [15], where a closed formula is given for the delta-invariant of X.

9. A classical identity of Thomae

In this final section we combine our Theorem 8.2 with a formula due to Guàrdia in order to obtain a symmetric version of an identity found in the 19th century by Thomae [23]. This identity relates a certain Jacobian Nullwert to a certain product of Thetanullwerte in the context of hyperelliptic period matrices. The classical proof of Thomae's identity can perhaps best be learnt from the paper [7] by Frobenius. Interestingly, in this classical proof the heat equation for the theta function plays a fundamental role. In our approach the heat equation is circumvented, which perhaps leads to a better "algebraic" understanding of Thomae's identity. We remark that the relations between Jacobian Nullwerte and Thetanullwerte have been studied extensively by Igusa, see for instance [12] and [13], and recently again by Guàrdia in his paper [9].

Let $g \geq 2$ be an integer. Let $\eta_1, \ldots, \eta_g$ be g odd theta characteristics in dimension g. We recall that the Jacobian Nullwert $J(\eta_1, \ldots, \eta_g)$ in $\eta_1, \ldots, \eta_g$ is defined to be the Jacobian

$$J(\eta_1, \ldots, \eta_g)(\tau) = \frac{\partial(\vartheta[\eta_1], \ldots, \vartheta[\eta_g])}{\partial(z_1, \ldots, z_g)}(0; \tau),$$

viewed as a function on $\mathcal{H}_g$, the Siegel upper half space. We want to study the values of Jacobian Nullwerte for period matrices coming from hyperelliptic Riemann surfaces. So let X be a hyperelliptic Riemann surface of genus g and let τ be a period matrix associated to a canonical symplectic basis of X, given by a certain ordering $W_1, \ldots, W_{2g+2}$ of its Weierstrass points. We recall from Section 5 that in this set-up, the Abel–Jacobi–Riemann map u induces a canonical bijection

$$\{\text{classes of semi-canonical divisors}\} \xrightarrow{\sim} \{\text{classes mod } \mathbb{Z}^g \times \mathbb{Z}^g \text{ of theta characteristics}\}$$

given by $[D] \mapsto [(\eta', \eta'')]$ if $u([D]) = [\eta' + \tau \cdot \eta'']$ on $J_\tau(X)$. Here we want to consider semi-canonical divisors of the form $W_{i_1} + \cdots + W_{i_{g-1}}$ for subsets $\{i_1, \ldots, i_{g-1}\}$ of $\{1, \ldots, 2g+2\}$ of cardinality $g-1$. Such divisors have h^0 equal to 1. Again, the remarkable point is that the theta characteristic corresponding to $W_{i_1} + \cdots + W_{i_{g-1}}$ depends only on the set $\{i_1, \ldots, i_{g-1}\}$, and not on X. We end up with a canonical map

$$\{\text{subsets } S \text{ of } \{1, \ldots, 2g+2\} \text{with } \#S = g-1\} \longrightarrow \{\text{classes mod } \mathbb{Z}^g \times \mathbb{Z}^g \text{of theta characteristics}\}.$$

One can prove that this map is 1-to-1, and that the theta characteristics in the image are always odd. Again, the correspondence can be made explicit; see again [22], Chapter IIIa for the details. Now choose a subset $\{i_1, \ldots, i_g\}$ of $\{1, \ldots, 2g+2\}$ of cardinality g, and for $k = 1, \ldots, g$ let η_k be the odd theta characteristic corresponding to $\{i_1, \ldots, \widehat{i_k}, \ldots, i_g\}$ by the above canonical map. We put

$$\|J\|(W_{i_1}, \ldots, W_{i_g}) = (\det \operatorname{Im} \tau)^{(g+2)/4} |J(\eta_1, \ldots, \eta_g)(\tau)|.$$

It can be checked that this only depends on the set $\{W_{i_1}, \ldots, W_{i_g}\}$ and not on the chosen ordering of the Weierstrass points. We have the following theorem.

Theorem 9.1 (*Thomae's Identity*). *Let X be a hyperelliptic Riemann surface of genus $g \geq 2$ with Weierstrass points $W_1, \ldots, W_{2g+2}$. Let $m = \binom{2g+2}{g}$. Then we have*

$$\prod_{\{i_1,\ldots,i_g\}} \|J\|(W_{i_1}, \ldots, W_{i_g}) = \pi^{gm} \|\varphi_g\|(X)^{(g+1)/4},$$

where the product runs over the subsets of $\{1, \ldots, 2g+2\}$ *of cardinality* g.

Our proof is basically a combination of Theorem 7.2 with the following proposition, which is a special case of the main theorem of [8]. The formula can be obtained from Faltings' formula (*) by a limiting process, using Riemann's singularity theorem.

Proposition 9.2 (*Guàrdia [8]*). *Let $W_{i_1}, \ldots, W_{i_g}, W$ be distinct Weierstrass points of X. Then the formula*

$$\|\vartheta\|(W_{i_1} + \cdots + W_{i_g} - W)^{g-1} = \mathrm{e}^{\delta(X)/8} \cdot \|J\|(W_{i_1}, \ldots, W_{i_g}) \cdot \frac{\prod_{k=1}^{g} G(W_{i_k}, W)^{g-1}}{\prod_{k<l} G(W_{i_k}, W_{i_l})}$$

holds.

Proof of Theorem 9.1. We start by taking a set $\{i_1, \ldots, i_g\}$ and taking the product over $W \notin \{W_{i_1}, \ldots, W_{i_g}\}$ in the formula from Proposition 9.2. This gives

$$\prod_{W \notin \{W_{i_1},\ldots,W_{i_g}\}} \prod_{k=1}^{g} G(W_{i_k}, W)^{2g-2}$$
$$= \mathrm{e}^{-(g+2)\delta(X)/4} \cdot \frac{\prod_{W \notin \{W_{i_1},\ldots,W_{i_g}\}} \|\vartheta\|(W_{i_1} + \cdots + W_{i_g} - W)^{2g-2}}{\|J\|(W_{i_1}, \ldots, W_{i_g})^{2g+4}} \cdot \prod_{k \neq l} G(W_{i_k}, W_{i_l})^{g+2}.$$

Taking the product over all sets $\{i_1, \dots, i_g\}$ of cardinality g we find

$$\prod_{(W,W')} G(W, W')^{n(g-1)}$$
$$= \mathrm{e}^{-m(g+2)\delta(X)/4} \cdot \prod_{\{i_1,\dots,i_g\}} \frac{\prod_{W \notin \{W_{i_1},\dots,W_{i_g}\}} \|\vartheta\|(W_{i_1} + \cdots + W_{i_g} - W)^{2g-2}}{\|J\|(W_{i_1}, \dots, W_{i_g})^{2g+4}}.$$

From our definition of $\|\varphi_g\|(X)$ it follows that

$$\|\varphi_g\|(X) = \prod_{\{i_1,\dots,i_{g+1}\}} \|\vartheta\|(W_{i_1} + \cdots + W_{i_g} - W_{i_{g+1}})^4,$$

where the product runs over the set of subsets of $\{1, 2, \dots, 2g+2\}$ of cardinality $g+1$. This gives

$$\prod_{\{i_1,\dots,i_g\}} \prod_{W \notin \{W_{i_1},\dots,W_{i_g}\}} \|\vartheta\|(W_{i_1} + \cdots + W_{i_g} - W)^{2g-2} = \|\varphi_g\|(X)^{(g^2-1)/2}.$$

Plugging this in our previous formula gives

$$\prod_{(W,W')} G(W, W')^{n(g-1)} = \mathrm{e}^{-m(g+2)\delta(X)/4} \cdot \|\varphi_g\|(X)^{(g^2-1)/2} \cdot \prod_{\{i_1,\dots,i_g\}} \|J\|(W_{i_1}, \dots, W_{i_g})^{-(2g+4)}.$$

Comparing this formula with the one in Theorem 8.2 gives the required formula. □

It is possible to derive from Theorem 9.1 a statement involving holomorphic functions on the domain of hyperelliptic period matrices in $\mathcal{H}_g$. We call a set $\{\eta_1, \dots, \eta_g\}$ of odd theta characteristics special if it can be obtained from a subset of $\{1, \dots, 2g+2\}$ of cardinality g in the way that we described above. Let H denote the set of special sets of odd theta characteristics, and let as before $\mathcal{S}$ denote the set of subsets of $\{1, \dots, 2g+2\}$ of cardinality $g+1$. Then one can deduce from our result that for period matrices τ associated to canonical symplectic bases of hyperelliptic Riemann surfaces of genus g one has

$$\prod_{\{\eta_1,\dots,\eta_g\} \in H} J(\eta_1, \dots, \eta_g)(\tau) = \pm\pi^{gm} \prod_{S \in \mathcal{S}} \vartheta[\eta_S](0; \tau)^{g+1}.$$

Indeed, one observes first that by dividing the left and right of the formula in Theorem 9.1 by an appropriate power of $\det \operatorname{Im} \tau$ one gets

$$\prod_{\{\eta_1,\dots,\eta_g\} \in H} |J(\eta_1, \dots, \eta_g)(\tau)| = \pi^{gm} |\varphi_g(\tau)|^{(g+1)/4}.$$

The maximum principle for holomorphic functions allows us then to write

$$\prod_{\{\eta_1,\dots,\eta_g\} \in H} J(\eta_1, \dots, \eta_g)(\tau) = \varepsilon\pi^{gm} \prod_{S \in \mathcal{S}} \vartheta[\eta_S](0; \tau)^{g+1},$$

where ε is a complex number of modulus 1 depending only on g. Considering the Fourier expansions on the left and right as in [12], pp. 86–88 one finds the value $\varepsilon = \pm 1$.

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